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THE PROPAGATION OF PERTURBATIONS IN SUPERSONIC BOUNDARY LAYERS†

I. I. LIPATOV

Zhukovskii

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Unsteady processes involving the propagation of perturbations in two-dimensional boundary layers are analysed in the strong interaction regime. A system of characteristics and subcharacteristics corresponding to wave processes in gas dynamics as well as convection and diffusion processes is determined. A system of equations describing the processes of weak interactions between the flow in a laminar boundary layer and an external supersonic flow near a cooled surface is analysed. For a flow described by the self-similar system of equations for the boundary layer, the velocity of propagation of perturbations as a function of a temperature factor is determined from a numerical solution. Copyright © 1996 Elsevier Science Ltd.

The propagation of perturbations in a boundary layer is related to convection and diffusion processes [1]. It has been shown by analysing the system of equations for a three-dimensional boundary layer [1] that normal lines to the surface placed in the flow serve as the characteristics of this system. The form of these characteristics and the conditions satisfied on them correspond to the propagation of perturbations with infinite velocity in a direction normal to the surface. These effects are related to diffusion processes, which occur in one direction in the system of equations for a boundary layer, which is degenerate relative to the original Navier–Stokes equations, and are determined by higher derivatives in the boundary-value problem. To describe the propagation of perturbations related to convection it is necessary to analyse the characteristics of the system of boundary-layer equations without higher derivatives (subcharacteristics). The complete system of characteristics and subcharacteristics enables qualitative conditions to be laid down for the boundary-value problem to be well-posed and enables the zone of dependence and influence to be determined. Next, the equations of a two-dimensional unsteady boundary layer were analysed and the system of characteristics and subcharacteristics was determined in [2]. The study of characteristics and subcharacteristics was determined in [2]. The study of characteristics and subcharacteristics in unsteady boundary layers containing cuspidal points led to the conclusion that discontinuous solutions may develop with time [3].

At the same time, because of the no-slip conditions on the surface and the presence of a domain of subsonic flow it is possible for perturbations to propagate due to wave processes. Experimental results concerning the effects of upstream propagation of perturbations in supersonic boundary layers are presented, for example, in [4]. Classical boundary-layer theory does not enable such processes to be described because it is assumed that the pressure distribution is known in advance. A mathematical model of linear processes of the interaction of a viscous flow in a boundary layer with an external inviscid flow was proposed in [5]. The effects of strong local viscous–inviscid interaction turn out to be significant in the case of upstream propagation of perturbations. When these effects are taken into account, it becomes possible to describe local flows with separation [6–8], as well as flows in domains with large local gradients [9–11].

An analysis of the propagation of perturbations in a three-dimensional boundary layer under steady interaction conditions enable the corresponding subcharacteristic surfaces [12] separating the domains of subcritical (subsonic on the average) and supercritical (supersonic on the average) flow to be determined in a hypersonic boundary layer near a triangular airfoil. The definition of sub- and supercritical flows is given in [13] for flows in which perturbations propagate upstream over distances comparable with the boundary-layer thickness or significantly exceeding the boundary-layer thickness.

Below we analyse unsteady flows in a laminar boundary layer under strong interaction conditions. Unsteady perturbations giving rise to viscous–inviscid interaction processes can be caused by time variations of the base pressure, by a shock wave of variable intensity hitting the boundary layer, etc. As has been shown in the study of hypersonic flows [14], upstream propagation over the whole surface up to the leading edge is characteristic for steady conditions of strong interactions. It is natural to assume

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that similar effects will also appear in unsteady flows, changing both the local and integral values of surface friction, heat flux and pressure. Modelling processes of this kind are therefore important for solving problems in practical aerodynamics.

1. We consider the flow past a flat surface (plate or wedge) at zero angle of attack to a free hypersonic stream of viscous heat-conducting gas. It is assumed that the strong interaction regime [15] occurs, for which the following relations are characteristic

$$M_\infty \to \infty, \quad M_\infty \tau \to \infty \tag{1.1}$$

where M_∞ is the Mach number of the free stream, and τ is the dimensionless thickness of a laminar boundary layer ($\tau = O(\mathrm{Re}_0^{-1/2})$). For Cartesian coordinates along the plate surface and normal to the surface, time, velocity vector components, density, pressure, total enthalpy, and dynamic coefficient of viscosity we introduce the notation lx, ly, lz, lt/u_∞, $u_\infty u$, $u_\infty v$, $\rho_\infty\rho$, $\rho_\infty u_\infty^2 p$, $u_\infty^2 H/2$, $\mu_0\mu$. The parameter l is either the characteristic length of the plate or the distance from the leading edge to the base of the wedge. The infinity subscript denotes the dimensional parameters of the inviscid flow over the boundary layer, while the zero subscript denotes the dimensional value of the dynamic coefficient of viscosity computed at the stagnation temperature. It is assumed that the Reynolds number $\mathrm{Re}_0 = \rho_\infty u_\infty l/\mu_0$ is large, but does not exceed the critical value at which the laminar regime can become turbulent. The Reynolds number is known to increase significantly as the Mach number increases [16].

In accordance with the theory of strong hypersonic interactions, the domain of perturbed flow near the surface placed in the flow is divided into two subdomains: 1—inviscid flow, 2—viscous flow (Fig. 1).

The following asymptotic representations of the stream functions and coordinates correspond to domain 1

$$x = x_1, \quad y = \tau y_1, \quad t = t_1$$
$$u(x,y,t,\tau) = 1+\ldots, \quad v(x,y,t,\tau) = \tau v_1(x_1,y_1,t_1)+\ldots$$
$$p(x,t,y,\tau) = \tau^2 p_1(x_1,y_1,t_1)+\ldots, \quad \rho(x,y,t,\tau) = \rho_1(x_1,y_1,t_1)+\ldots \tag{1.2}$$

Substituting expansions (1.2) into the system of Navier–Stokes equations and taking the limit (1.1) we obtain a system of equations of the form

$$\frac{\partial \rho_1}{\partial t_1} + \frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1 v_1}{\partial y_1} = 0$$

$$\frac{\partial v_1}{\partial t_1} + \frac{\partial v_1}{\partial x_1} + v_1\frac{\partial v_1}{\partial y_1} + \frac{1}{\rho_1}\frac{\partial p_1}{\partial y_1} = 0, \quad \frac{\partial}{\partial x_1}\left(\frac{p_1}{\rho_1^\gamma}\right) + v_1\frac{\partial}{\partial y_1}\left(\frac{p_1}{\rho_1^\gamma}\right) = 0$$

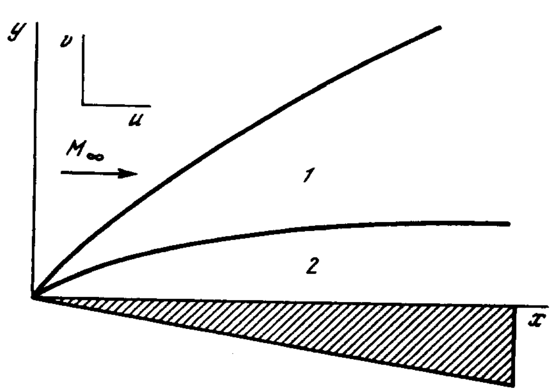

Fig. 1.

with the following boundary conditions on the shock wave

$$y_1 = g_1(x_1,t_1), \quad \rho_1 = \frac{(\gamma+1)}{(\gamma-1)}, \quad p_1 = \frac{(\gamma+1)\, v_1^2}{2}, \quad v_1 = \frac{2}{(\gamma+1)}\left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_1}{\partial t_1}\right)$$

and on the outer boundary of the boundary layer

$$y_1 = \delta_1(x_1,t_1), \quad v_1 = \frac{2}{(\gamma+1)}\left(\frac{\partial \delta_1}{\partial x_1} + \frac{\partial \delta_1}{\partial t_1}\right)$$

For the analysis below it is necessary to obtain a relation between the thickness δ_1 of the boundary layer or the vertical velocity $v_1(x_1, \delta_1, t_1)$ and the pressure perturbation $p_1(x_1, t_1)$. Below we use the approximate relation

$$p_1 = (\gamma+1)\, v_1^2/2 \tag{1.3}$$

which is an extension of the tangent wedge formula to the unsteady case.

The following asymptotic expansions and representations of coordinates are characteristic for domain 2

$$\begin{gathered} x = x_1, \quad y = \tau y_1, \quad t = t_1 \\ u(x,y,t,\tau) = u_2(x_1,y_1,t_1)+\ldots, \quad v(x,y,t,\tau) = \tau v_2(x_1,y_1,t_1)+\ldots \\ p(x,y,t,\tau) = \tau^2 p_2(x_1,t_1)+\ldots, \quad \rho(x,y,t,\tau) = \tau^2 \rho_2(x_1,y_1,t_1)+\ldots \\ H(x,y,t,\tau) = H_2(x_1,y_1,t_1)+\ldots \end{gathered} \tag{1.4}$$

Substituting expansions (1.4) into the system of Navier–Stokes equations and taking the limit (1.1), we obtain a system of equations for an unsteady boundary layer. The replacement of variables

$$X = x_1, \quad T = t_1, \quad Y = \left[\frac{2\gamma C_0}{(\gamma-1)}\right]^{-1/2} x_1^{-1/4}\int_0^{y_1} R\,dy_1, \quad u_2 = \frac{\partial F}{\partial Y}$$

$$p_2 = x_1^{-1/2}P, \quad \rho_2 = x_1^{-1/2}R, \quad C_0 = P_{X=0}, \quad G = H_2, \quad A = G - U^2$$

reduces the corresponding boundary-value problem to the form

$$\begin{gathered} X\frac{\partial U}{\partial T} + X\left(U\frac{\partial U}{\partial X} - \frac{\partial F}{\partial X}\frac{\partial U}{\partial Y}\right) - \frac{F}{4}\frac{\partial U}{\partial Y} + \beta\frac{(\gamma-1)}{4\gamma}A = \frac{P}{C_0}\frac{\partial^2 U}{\partial Y^2} \\ X\frac{\partial G}{\partial T} + X\left(U\frac{\partial G}{\partial X} - \frac{\partial F}{\partial X}\frac{\partial G}{\partial Y}\right) - \frac{F}{4}\frac{\partial G}{\partial Y} = XA\frac{(\gamma-1)}{\gamma P}\frac{\partial P}{\partial T} + \frac{P}{C_0}\frac{\partial^2 G}{\partial Y^2} \\ \beta = -1 + \frac{2X}{P}\frac{\partial P}{\partial X}, \quad \Delta = \left[\frac{(\gamma-1)C_0}{2\gamma P^2}\right]^{1/2}\int_0^\infty A\,dY, \quad P = \frac{(\gamma+1)}{2}\left[\frac{3\Delta}{4} + X\left(\frac{\partial\Delta}{\partial X} + \frac{\partial\Delta}{\partial T}\right)\right]^2 \\ U = F = 0, \quad G = g_w, \quad Y = 0; \quad U = G = 1, \quad Y = \infty \\ P(1,T) = \varphi(T) \end{gathered} \tag{1.5}$$

where it is assumed that the dynamic coefficient of viscosity depends linearly on temperature, the Prandtl number is equal to one, and the last boundary condition corresponds to the given time dependence of the base pressure drop.

We will first determine the characteristic (subcharacteristic) surfaces $\Omega(X, T)$ related to the function $P(X, T)$, which is unknown in advance and can be determined in the course of solving the problem.

After making the change of variables

$$X, Y, T \to \Omega, Y, T \tag{1.6}$$

the boundary-value problem (1.5) takes the form

$$b\left(S\frac{\partial U}{\partial \Omega}-\frac{\partial F}{\partial \Omega}\ \frac{\partial U}{\partial Y}+CA\frac{\partial P}{\partial \Omega}\right)=B,\quad b\left(S\frac{\partial G}{\partial \Omega}-\frac{\partial F}{\partial \Omega}\ \frac{\partial G}{\partial Y}-2CAa\frac{\partial P}{\partial \Omega}\right)=D \tag{1.7}$$

Here

$$S=U+a,\quad a=\frac{\partial \Omega}{\partial T}\left(\frac{\partial \Omega}{\partial X}\right)^{-1},\quad b=X\frac{\partial \Omega}{\partial X},\quad B=\left[\frac{\partial^2 U}{\partial Y^2}+\frac{F}{4}\ \frac{\partial U}{\partial Y}+\frac{(\gamma-1)A}{4\gamma}-X\frac{\partial U}{\partial T}\right]$$

$$C=\frac{(\gamma-1)}{2\gamma P},\quad D=\left[\frac{\partial^2 G}{\partial Y^2}+\frac{F}{4}\ \frac{\partial G}{\partial Y}+2XCA\frac{\partial P}{\partial T}-X\frac{\partial G}{\partial T}\right]$$

The interaction condition, relating the pressure distribution with the boundary-layer displacement thickness, can be transformed as follows:

$$b(1+a)\frac{\partial \Delta}{\partial \Omega}=c\left(c=(2P)^{1/2}(\gamma-1)^{-1/2}-3\frac{\Delta}{4}-X\frac{\partial \Delta}{\partial T}\right) \tag{1.8}$$

The derivative on the left-hand side of (1.8) can be expressed in accordance with the above expression for the displacement thickness

$$\frac{\partial \Delta}{\partial \Omega}=\left(\frac{\gamma-1}{2\gamma}\ \frac{C_0}{P^2}\right)^{1/2}\left[\int_0^\infty\left(\frac{\partial G}{\partial \Omega}-2U\frac{\partial U}{\partial \Omega}\right)dY-\frac{1}{P}\ \frac{\partial P}{\partial \Omega}\int_0^\infty AdY\right]$$

To compute the derivatives with respect to Ω in the integrands one can use (1.5), from which it follows that

$$\frac{\partial F}{\partial \Omega}=-\frac{S}{P}\ \frac{\partial P}{\partial \Omega}+\int_0^Y \frac{A}{S^2}dY+S\int_0^Y \frac{B}{S^2}dY,\quad \frac{\partial G}{\partial \Omega}=\frac{1}{S}\ \frac{\partial F}{\partial \Omega}\ \frac{\partial G}{\partial Y}+\frac{D}{S}+\frac{2CAaT}{S}\ \frac{\partial P}{\partial \Omega}$$

After some reduction using relations (1.7)–(1.8), we obtain the expression

$$bN=\frac{\partial P}{\partial \Omega}=PM \tag{1.9}$$

$$M=\int_0^\infty BdY-S^2\int_0^\infty \frac{B}{S^2}dY-\int_0^\infty \frac{D}{S}dY-\left(\frac{\gamma-1}{2\gamma}C_0\right)^{-1/2}c,\quad N=\int_0^\infty \frac{A^2}{S^2}dY-\frac{2}{(\gamma-1)}\int_0^\infty AdY$$

The condition defining the subcharacteristic surface has the form

$$\frac{(\gamma-1)}{2}\int_0^\infty \frac{(G-U^2)^2}{(U+a)^2}dY-\int_0^\infty (G-U^2)dY=0 \tag{1.10}$$

and is an extension of Pearson's integral [17]

$$L=\frac{(\gamma-1)}{2}\int_0^\infty \frac{(G-U^2)^2}{U^2}dY-\int_0^\infty (G-U^2)dY \tag{1.11}$$

the sign of which depends on the mean value of the Mach number in the boundary layer. A negative value of the integral corresponds to supersonic flow on the average, while a positive value corresponds to a flow that is subsonic on the average.

Relation (1.10) has a simple physical meaning. The average flow velocity over the profile exists in a hypersonic boundary layer. This being so, if the average velocity of sound is greater than the average flow velocity, then the flow in the boundary layer is subcritical and perturbations propagate upstream. Correspondingly, the flow will be supercritical if the average velocity is less than the velocity of sound.

Formula (1.10) can be obtained from (1.11) by a simpler method. We change from the stationary system of coordinates X, T to a system X_1, T moving upstream at constant velocity $X_1 = X + aT$.

In the moving system of coordinates the velocity in the boundary layer is equal to $U_1 = U + a$ and the difference $G_1 - U^2_1 = G - U^2$ does not change because it is proportional to the gas temperature. Substituting the expressions for U_1 and G_1 into (1.1), we obtain (1.10) with U_1, G_1 in place of U, G.

According to the above definitions

$$a = (\partial\Omega / \partial X)(\partial\Omega / \partial T)^{-1} = -dX / dT$$

A sound wave propagating upstream (downstream) corresponds to $a > 0$ ($a < 0$). The first integral in (1.10) converges when $a < 0$, since $| a > 1 |$.

As an example, we shall present the dependence of the upstream and downstream wave propagation velocities on the temperature factor obtained when solving the self-similar system of equations obtained from (1.5). These functions are presented in Fig. 2. It can be seen that the velocity of upstream propagation of perturbations tends to zero as the temperature factor decreases, which means passing to supercritical conditions.

Relation (1.10) can also be obtained for other forms of flow. An analysis of (1.7)–(1.9) shows that (1.10) also holds for generalized local conditions for the relation between the induced pressure and displacement thickness of the form $P = \Gamma(\Delta, d\Delta/dX)$, including the conditions for flow in a plane-parallel channel (Δ = const).

It should also be noted that the velocity profiles and enthalpies in (1.10) may also correspond to turbulent flows.

2. The mutual influence of viscous and inviscid flows studied above corresponds to a global (strong) interaction, which appears in the limit (1.1) over the whole length of the body. It is also known that in the limit as

$$M_\infty \to \infty, \quad M_\infty \tau_1 \to 0 \tag{2.1}$$

where τ_1 is the dimensionless boundary-layer thickness ($\tau_1 = O(\varepsilon_0)$, $\varepsilon_0 = \rho_0 u_\infty l/\mu_0$), strong interaction processes, which appear, for example, under the influence of base pressure variations or other causes,

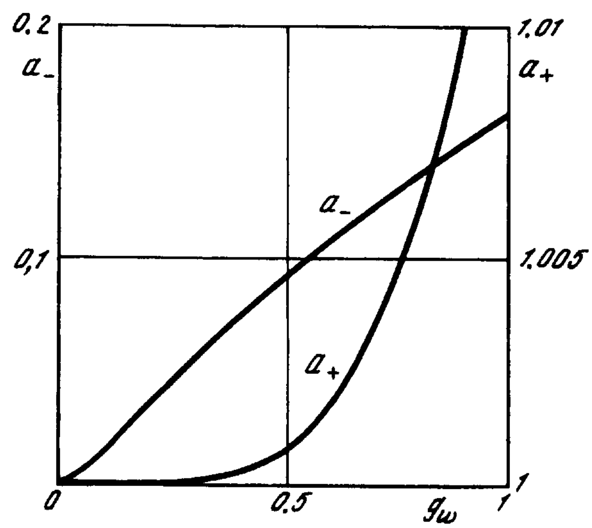

Fig. 2.

have a local character and are present at distances asymptotically small compared with the body length. Thus, in the limit (2.1), subject to the relations

$$g_w \to 0, \quad \varepsilon_0 g_w^2 M_\infty = O(1), \quad \Delta p M_\infty^{3/2} g_w^{-\omega/2} \varepsilon_0^{-1/2} = O(1)$$

(where ω is the exponent in the exponential dependence of the viscosity on the temperature) the boundary-value problem describing the perturbed flow has the form [18, 19]

$$\begin{aligned}
&\frac{\partial U_0}{\partial T_0} + U_0\frac{\partial U_0}{\partial X_0} + V_0\frac{\partial U_0}{\partial Y_0} + \frac{\partial P_0}{\partial X_0} = \frac{\partial^2 U_0}{\partial Y_0^2}\\
&\frac{\partial U_0}{\partial X_0} + \frac{\partial V_0}{\partial Y_0} = 0, \quad P_0 = -\frac{\partial D_0}{\partial X_0} + N_0\frac{\partial P_0}{\partial X_0}\\
&X_0 \to -\infty, \quad U_0(X_0, Y_0, T_0) = Y_0\\
&Y_0 \to \infty, \quad U_0(X_0, Y_0, T_0) = Y_0 + D_0(X_0, T_0) + o(1)\\
&P_0 = \Delta p(2a_0 g_w^{\omega} w_0 M_\infty^{-3}(\gamma-1)^{-1})^{-1/2}\\
&X_0 = (x-1)(2^3 a_0^5 g_w^{(\omega-2)} \varepsilon_0^{-3} M_\infty^{-3}(\gamma-1)^{-3})^{1/4}\\
&Y_0 = y(2a_0^3 g_w^{-(2+\omega)} \varepsilon_0^{-5} M_\infty^{-1}(\gamma-1)^{-1})^{1/4}\\
&U_0 = u(2^{-1} a_0 g_w^{(2+\omega)} \varepsilon_0 M_\infty(\gamma-1))^{-1/4}\\
&N_0 = (d\tau_1/dp)(2^3 a_0^5 g_w^{(\omega-2)} \varepsilon_0^{-3} M_\infty^{-7}(\gamma-1)^{-3})^{1/4}
\end{aligned} \tag{2.2}$$

where the parameter a_0 is proportional to the friction stress in the boundary layer in front of the interaction domain ($a_0 = \varepsilon_0(\partial u/\partial y)_w$).

If we change to the new variables

$$X_0, Y_0, T_0 \to \Omega_0(X_0, T_0), Y_0, T_0$$

in the boundary-value problem (2.2), then, after some reductions we can obtain the following expression

$$\frac{\partial P_0}{\partial \Omega_0} = \frac{M_1}{N_1}, \quad M_1 = I_1 + P_0\left(\frac{\partial \Omega_0}{\partial X_0}\right)^{-1}, \quad N_1 = N_0 + I_1$$

$$I_1 = \int_0^\infty \frac{dY_0}{(a_1 + U_0)^2}, \quad a_1 = \left(\frac{\partial \Omega_0}{\partial X_0}\right)\left(\frac{\partial \Omega_0}{\partial T_0}\right)^{-1}$$

In the case of small amplitude perturbations, when $a_1 + U_0 \approx a_1 + Y_0$, we have $a_1 = -1/N_0$.

3. Consider the system of characteristics and subcharacteristics for all the functions appearing in the boundary-value problem (1.5). If the pressure $P(X, T)$ is assumed to be given in this problem, the change of variables (1.6) enables us to obtain the following relation after some reduction

$$A_0 \partial \Phi_0 / \partial \Omega = B_0, \quad A_0 = \mathrm{diag}(z^2, z^2, z), \quad \Phi_0 = \mathrm{col}(U, G, F), \quad z = \partial \Omega / \partial Y$$

According to the definition of a characteristic, from the condition $\det A_0 = 0$ one can obtain $(\partial\Omega/\partial Y)^5 = 0$. It follows that the lines (X, T) = const normal to the surface are the characteristics of the system of equations for an unsteady boundary layer. These characteristics are related to diffusion processes occurring normal to the surface within the flow, which give rise to the propagation of perturbations at infinite velocity.

To analyse the propagation of perturbations in planes parallel to the surface placed in the flow, which are described by first-order differential operators, it is necessary to consider the system of equations for an unsteady boundary layer without higher derivatives.

The replacement of variables (1.6) in the corresponding system of equations obtained from (1.5) yields

$$A_1 \partial\Phi_1 / \partial\Omega = B_1$$

$$A_1 = \left\| \begin{matrix} \Delta^* & 0 & c_1 \\ 0 & \Delta^* & c_1 \\ 0 & 0 & z \end{matrix} \right\|, \quad \Phi_1 = \left\| \begin{matrix} U \\ G \\ F \end{matrix} \right\|, \quad \Delta^* + \frac{\partial\Omega}{\partial T} + U\frac{\partial\Omega}{\partial X} - \frac{\partial F}{\partial X}\frac{\partial\Omega}{\partial Y}, \quad c_1 = -\frac{\partial\Omega}{\partial X}\frac{\partial U}{\partial Y}$$

The characteristics corresponding to this problem or the subcharacteristics of the original problem can be determined from the condition

$$\Delta^{*2} \partial\Omega / \partial Y = 0 \tag{3.1}$$

The analysis of an analogue of this relation derived for incompressible flow [2] enables us to determine the zones of dependence and influence. It turned out that the local zone of influence on the flow in the neighbourhood of the line (X_1, T_1) = const is bounded by surfaces whose projections onto the plane $Y = 0$ are determined by the maximum and minimum values of the derivative $dX/dT = U(X_1, Y, T_1)$.

We will consider the boundary-value problem (1.5) describing the flow in an unsteady boundary layer under strong interaction conditions. Changing to the variables (1.6) and using (1.9) to determine the derivative $\partial P/\partial\Omega$, we find that

$$A_2 = \left\| \begin{matrix} \Delta^* & 0 & 0 & c_2 \\ 0 & \Delta^* & 0 & c_3 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & c_4 \end{matrix} \right\|, \quad \Phi_2 = \left\| \begin{matrix} U \\ G \\ F \\ P \end{matrix} \right\|$$

$$c_2 = \frac{X(\gamma-1)}{2\gamma P}\frac{\partial\Omega}{\partial X}, \quad c_3 = -\frac{X(\gamma-1)(G-U^2)}{\gamma P}\frac{\partial\Omega}{\partial T}, \quad c_4 = XN\frac{\partial\Omega}{\partial X}$$

The relation defining the subcharacteristics takes the form

$$N\frac{\partial\Omega}{\partial X}\frac{\partial\Omega}{\partial Y}\left(\frac{\partial\Omega}{\partial T} + U\frac{\partial\Omega}{\partial X} - \frac{\partial F}{\partial X}\frac{\partial\Omega}{\partial Y}\right)^2 = 0 \tag{3.2}$$

where the additional term as compared to (3.1) corresponds to gas-dynamical wave processes.

The relations presented above and the zones of influence and dependence determined from them must be taken into account both when formulating boundary-value problems for the system of equations of an unsteady boundary layer under strong interaction conditions and when constructing the corresponding difference schemes. Formula (3.2) can be extended to the case of a three-dimensional unsteady interacting boundary layer.

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